Definition and Uniqueness of Integral Approximants

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Abstract: Definitions are given for the integral approximant polynomials which insure their existence and uniqueness. A specification of minimality is required in these definitions. Existence of infinite subsequences of integral polynomials without common factors of z is proven. An equivalence theorem between the property of agreement of all high order integral polynomials and the property that the function being approximated belongs to a particular function class is proved. The accuracy-through-order property is found to hold for all the cases we have investigated for the integral approximant. An example is given which proves that the series coefficients which uniquely determine the integral polynomials may not uniquely determine the integral approximant.

Keywords: Hermite-Padé approximant, integral approximant, differential approximant, Padé approximant, differential equations.

1. Introduction and Summary

Integral approximants are a special case of Hermite-Padé approximants (Hermite [21], Padé [29]) which were introduced to modern applications by Guttmann and Joyce [19-20], Gammel [17], Guttmann [18], Hunter and Baker [22], Fisher and Au-Yang [16], and Rehr *et al.* [30]. Much has been written about their theory recently, *e.g.* Della Dora and Di Crescenzo [15], Della Dora [13-14], Burley *et al.* [11], Baumel *et al.* [10], Baker [2-5] Nuttall [26-27], Baker and Lubinsky [8], Stahl [31], Baker *et al.* [9], and Baker and Graves-Morris [7]. Nevertheless, so far as the authors are aware, some very basic properties of integral approximants have not been studied to date. In the case of Padé approximants, (a special case of integral approximants) properties of these types have been so well studied (*e.g.* Baker and Graves-Morris [6]) that in the minds of many of those who apply them, they are considered as obvious truisms.

As background for our studies of the integral approximants, we will now review some of these properties for Padé approximants. First, the modern definition in terms of a given function $f(z)$ defined by a Taylor series at $z=0$ is given in terms of two polynomials $P_L(z)$ and $Q_M(z)$ of degrees $L,\ M$ respectively, which in turn are defined by the set of linear equations,

$$Q_M(z)f(z) - P_L(z) = O(z^{L+M+1}),\ Q_M(0) = 1. \tag{1.1}$$

The approximant is then

$$[L/M]_f(z) = \frac{P_L(z)}{Q_M(z)}. \tag{1.2}$$

The approximant satisfies the same accuracy through order property as was present in the polynomial defining equations, namely,

$$f - [L/M] = O(z^{L+M+1}), \tag{1.3}$$

by virtue of the normalization in (1.1).

The next basic property is uniqueness (Frobenius [17], Padé [28]). Suppose that there are two solutions of (1.1), X_L/Y_M and P_L/Q_M, then by subtraction of (1.3) for each of them and cross multiplication,

$$Y_M P_L - Q_M X_L = O(z^{L+M+1}), \tag{1.4}$$

but the polynomial on the left-hand side of (1.4) is of maximum degree $L+M$ and so is identically zero. Thus, the $[L/M]$ is unique.

The next property is equivalence (Padé [28], and proven with the modern definition by Chisholm [12]). For a function $g(z)$ which is analytic at $z=0$, the statement is

$$g(z) = \frac{\sum_{t=0}^{l} c_t z^t}{1+\sum_{t=1}^{m} e_t z^t}, \tag{1.5}$$

if and only if,

$$[L/M]_g = [l/m]_g\ \ \forall\, L \geq l,\ M \geq m. \tag{1.6}$$

Finally, required to prove convergence theorems, there is the existence of at least infinite subsequences of specified type (Baker [1], Magnus [25]). In particular it has been proven that there exists at least an infinite subsequence of the types $[N/N]$, $[N/m]$ and $[l/N]$ as $N \to \infty$ with l, m fixed. In fact these basic properties are required even to begin a detailed discussion of Padé approximant theory.

In the second section we show that there does exist a definition, in terms of what we call the *minimality* property, of the integral polynomials (which in their turn will be used to define the integral approximant as the Padé polynomials were used above to define the Padé approximant) which insures that, for each assignment of the degrees of the integral polynomials, the integral polynomials both exist and are essentially unique. We also prove without the imposition of the minimality property that, if the differential equation defining the integral approximant is of order k, then there are at most $k+1$ linearly independent solutions of the integral polynomial defining equations.

An example for Padé approximants shows what section three is about. The [1/1] Padé approximant to the function $f(z) = 1 + z^2 + O(z^3)$, without the requirement $Q_1(0) = 1$ is

$$P_1(z) = z,\ Q_1(z) = z,\ P_1/Q_1 = 1, \tag{1.7}$$

which fails to maintain the accuracy through order property, while if the requirement $Q_1(0) = 1$ is maintained, there is no [1/1]. In the third section we show that in fact there exists at least an infinite subsequence of diagonal (all the integral polynomials are of the same nominal degree) integral polynomials which have the property that they do not have z as a common divisor. We further show that for the more general case of any connected, increasing sequence that it has at least an infinite subsequence for which the integral polynomials do not have z as a common divisor. As a special case, the same is true of partially diagonal sequences (*i.e.* when all the polynomial degrees which go to infinity with the sequence index are equal to each other.) Regions in the index space describing the integral polynomials which contain solutions of the integral polynomial defining equations reducible to a single fundamental solution are shown to include the union of certain simplices.

The fourth section is devoted to proving an equivalence theorem for integral approximants. The class of functions which takes the place of rational fractions (1.5) for integral approximants is the class of functions which are the solutions of non-homogeneous, linear, ordinary differential equations of specific order with polynomial coefficients. The fact of agreement of all higher order integral polynomials is shown to be equivalent to the function which is being approximanted belonging to this class. An example is given which shows that the number of series terms required to uniquely determine the integral polynomials may not be sufficient to also uniquely determine the integral approximant.

In the last section we discuss the accuracy-through-order property for integral approximants. We find that it is maintained for the approximants for the cases that we have analyzed.

2. Definition of Minimal Integral Polynomials

First we introduce some notation. Suppose that k and L are non-negative integers, and that $m_0, \ldots, m_k$ are integers which are greater than or equal to -1. Then let

$$\vec{m} = (m_0, m_1, \ldots, m_k),\quad M = \sum_{j=0}^{k} (m_j + 1) - 1. \tag{2.1}$$

P_L will denote a polynomial of degree at most L, and we write

$$\vec{Q}_L = (Q_{0,L}, Q_{1,L}, \ldots, Q_{k,L}) \tag{2.2}$$

to denote a vector of polynomials $Q_{j,L}$ of degree $\vec{m}$, and we will suppose that $\vec{Q}_L \not\equiv 0$.

Further let us be given a functional element $f(z) \not\equiv 0$ (By a functional element we mean a partial representation of a locally analytic function $f(z)$, in this case, a Taylor series about $z = 0$ convergent in a neighborhood.), then let

$$\sum_{j=0}^{k} f^{(j)}(z)Q_{j,L}(z) - P_L(z) = O(z^{L+M+1}). \tag{2.3}$$

The $\vec{Q}_L$, P_L are called integral polynomials of type $(L,\vec{m})$ to $f(z)$. The integral approximant of type $(L,\vec{m})$ is denoted by $[L/\vec{m}]$, which is short for $[L/m_0;\dots;m_k]$ in the more customary notation and is the solution $y(z)$ of

$$\sum_{j=0}^{k} y^{(j)}(z)Q_{j,L}(z) - P_L(z) = 0, \tag{2.4}$$

subject to the boundary conditions (usually but as we will see later not always!)

$$y^{(j)}(0) = f^{(j)}(0),\ j = 0,\dots,k-1. \tag{2.5}$$

Now the defining equation (2.3) for the integral polynomials is a system of M, linear, homogeneous equations for the $M+1$ coefficients of the $Q_{j,L}(z)$. This system has one more unknown than it has equations so by the standard theory of equations, there must exist a non-trivial solution for the $\vec{Q}_L$, (i.e. $\vec{Q}_L \not\equiv 0$).

This solution may or may not be essentially unique. Since the equations are homogeneous, if $\vec{Q}_L$ is a solution, then so too is $c\vec{Q}_L$ for any constant c. If the rank of the coefficient matrix is $r < M$, then there is an $M+1-r$ parameter family of solutions. However, we may prove the following multi-niqueness theorem.

Theorem 2.1. *There exist at most $k+1$ linearly independent solutions $(P_L^{(i)}, \vec{Q}_L^{(i)})$ for the integral polynomials of type $(L,\vec{m})$ to $f(z)$.*

Proof. Suppose that there exist $k+2$ solutions of the equations for the integral polynomials. Then

$$\sum_{j=0}^{k} f^{(j)}(z)Q_{j,L}^{(i)}(z) - P_L^{(i)}(z) = r_L^{(i)}(z),\ i = 1,\dots,k+2 \tag{2.6}$$

where for all i,

$$r_L^{(i)}(z) = O(z^{L+M+1}). \tag{2.7}$$

Regarding (2.6) as a set of linear equations in the $f^{(j)}(z),\ j = 0,\dots,k$, we have the condition,

$$\det\begin{vmatrix} Q_{k,L}^{(1)} & \cdots & Q_{0,L}^{(1)} & -P_L^{(1)} \\ \vdots & \ddots & \vdots & \vdots \\ Q_{k,L}^{(k+2)} & \cdots & Q_{0,L}^{(k+2)} & -P_L^{(k+2)} \end{vmatrix} = \det\begin{vmatrix} Q_{k,L}^{(1)} & \cdots & Q_{0,L}^{(1)} & r_L^{(1)} \\ \vdots & \ddots & \vdots & \vdots \\ Q_{k,L}^{(k+2)} & \cdots & Q_{0,L}^{(k+2)} & r_L^{(k+2)} \end{vmatrix}, \tag{2.8}$$

for their internal consistency. Since every term on the left-hand side of (2.8) is a polynomial of degree $\leq M-k+L$ and the right hand side is $O(z^{M+L+1})$, the left hand side must be identically zero and therefore any $k+2$ solutions must be linearly dependent. ■

Note: This result is actually of wider applicability to more general Hermite-Padé approximants.

In the cases where $M+1-r > k+1$, the extra solutions can be constructed as a linear combination of some polynomial factors $\mathcal{A}^{(i)}(z)$ times members of the basis set of at most $k+1$ solutions. Some members of the basis set will necessarily be of degree less than $(L,\vec{m})$ to permit the product still to be of type $(L,\vec{m})$. These results suggest the definition.

Definition 2.2. *A solution for the integral polynomials of type $(L,\vec{m})$ to $f(z)$ is called minimal if it is of the lowest degree in the following sense. First there exists no other solution of type $(L,\vec{m})$ for which the actual degree of $Q_{k,L}$ is smaller. If there exist solutions of type $(L,\vec{m})$ for which $Q_{k,L}\equiv 0$, then we minimize the degree of $Q_{k-1,L}, Q_{k-2,L}$, etc. to find the minimal solution.*

Theorem 2.3, Uniqueness. *For any type $(L,\vec{m})$ there exist essentially unique integral polynomials $(P_L,\vec{Q}_L)$ to $f(z)$ which are minimal.*

Proof. First we saw before that a solution exists. Second, suppose that there exist two of minimal degree where $Q^{(i)}_{j,L}(z)$ is of true degree $n_j\le m_j,\ i=1,2$, and $j\le k$ is the largest index for which the $Q^{(i)}_{j,L}\not\equiv 0$. Then

$$\alpha(P^{(1)}_L,\vec{Q}^{(1)}_L)+(1-\alpha)(P^{(2)}_L,\vec{Q}^{(2)}_L), \tag{2.9}$$

is also a solution of minimal degree. Choose α such that the coefficient of z^{n_j} in that solution is zero. But this result implies that n_j is not the true minimal degree, contrary to assumption, which is a contradiction. Therefore there is only one linearly independent, minimal solution. ∎

If $Q_{k,L}(0)\neq 0$, when we use the normalization for $Q_{k,L}(z)$,

$$Q_{k,L}(z)=\det\left|\begin{array}{cccc|c|ccc} f_{L+1} & f_L & \cdots & f_{L+1-m_0} & \cdots & f^{(k)}_{L+1} & \cdots & f^{(k)}_{L+1-m_k}\\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots\\ f_{M+L} & f_{M+L-1} & \cdots & f_{M+L-m_0} & \cdots & f^{(k)}_{M+L} & \cdots & f^{(k)}_{L+M-m_k}\\ 0 & 0 & \cdots & 0 & 0\cdots 0 & 1 & \cdots & z^{m_k}\end{array}\right|, \tag{2.10}$$

then we could have concluded a unique solution for the integral polynomials without any extra discussion. Can we hope to impose this condition in general on the integral polynomials? For the Padé approximants it made sense because we were given a function to approximate which was finite at $z=0$. Consider the example

$$\begin{aligned} f(z) &= \frac{2}{z}(1-\sqrt{1-z})+z^5\\ &= 1+\frac{1}{4}z+\frac{1}{8}z^2+\frac{5}{64}z^3+\frac{7}{128}z^4+\frac{533}{512}z^5+\cdots, \end{aligned} \tag{2.11}$$

which is regular at $z=0$. For type $L=0,\ m_0=1,\ m_1=2$ we have

$$z(1-z)y'+(1-\frac{1}{2}z)y=1. \tag{2.12}$$

The solution of (2.12) is

$$y(z)=\frac{2}{z}+\frac{B\sqrt{1-z}}{z}. \tag{2.13}$$

Here the boundary condition that $y(0)<\infty$ requires that $B=-2$, and implies that $y=f-z^5$. However this example is singular on the second Riemann sheet at $z=0$. Therefore it is required that $Q_{1,0}(0)=0$ in order to represent this singularity on that sheet. In this way the integral approximants are different from the Padé approximants in that the defining equation must reflect the behavior on other Riemann sheets than the first one and may well show singularities that are entirely absent on the first Riemann sheet.

3. Existence of Infinite Sequences of Integral Polynomials

Manifestly, since in the previous section we have shown that for all types $(L,\vec{m})$ the minimal integral polynomials exist and are unique, any type of sequence must exist. In this section however we will be concerned with the possibility of requiring some additional (desirable) properties. Clearly since the division by a common factor would lower the degree of the leading polynomial, the minimality condition serves to preclude the integral polynomials from having common factors of the form $1+az$. There may however be common factors which are of the form z^j. They can not be so simply divided out because to do so would change the term $O(z^{L+M+1})$ on the right-hand side of the defining equations (2.3).

Let us now consider the consequences of dividing out a common factor of z^j. First we shall look at a diagonal sequence of integral approximants, $[N/N;\ldots;N]$, *i.e.* $L=N$, $m_i=N$, $i=0,\ldots,k$ or $(L,\vec{m})=(N,\vec{N})$. We have seen that there exists a minimal solution for each N, such that

$$Q_{k,N}(z)f^{(k)}(z)+\cdots+Q_{0,N}(z)f(z)-P_N(z)=O\left(z^{(N+1)(k+2)-1}\right). \tag{3.1}$$

Now assume that $P,\vec{Q}$ is divisible by z^j so that (3.1) can be rewritten as

$$\hat{Q}_{k,N-j}(z)f^{(k)}(z)+\cdots+\hat{Q}_{0,N-j}(z)f(z)-\hat{P}_{N-j}(z)=O\left(z^{(N-j+1)(k+2)-1+j(k+1)}\right), \tag{3.2}$$

where $\hat{Q}_{i,N-j}\equiv z^{-j}Q_{i,N}$, $\hat{P}_{N-j}\equiv z^{-j}P_N$. By inspection of the error term on the right-hand side of (3.2) it is clear that it determines the integral polynomials of the type $(N-j,\overrightarrow{N-j})$, with a factor of $z^{j(k+1)}$ to spare. That the $\hat{Q}$'s and $\hat{P}$ are also minimal can be seen in the following way. Eq. (3.1) implies that

$$[Q_{k,N}(z)f^{(k)}(z)+\cdots+Q_{0,N}(z)f(z)]_n^n=0,\ n=L+1,\ldots,L+M, \tag{3.3}$$

where $[\,]_n^m$ is the section of the power series running from terms of order z^n to order z^m. Thus we must have the same set of equations for the $(N-j,\vec{N})$ integral polynomials,

$$[\hat{Q}_{k,N-j}(z)f^{(k)}(z)+\cdots+\hat{Q}_{0,N-j}(z)f(z)]_n^n=0,\ n=L-j+1,\ldots,L-j+M. \tag{3.4}$$

Since the Q's were minimal, so too are the $\hat{Q}$'s, provided that we restrict their degrees to be less than or equal to $N-j$. When this is done, we find that by having taken advantage of our knowledge of the minimal solution of (3.3), we can discard the last $j(k+1)$ equations of (3.4) as automatically over-satisfied and then by the identity of the remaining equation of (3.4) with those of (3.3), we conclude that the $\hat{Q}$'s are the minimal integral polynomials of $(N-j,\overrightarrow{N-j})$ type.

The most that the error term in (3.1) can be reduced by this procedure, since we know $\vec{Q}_L\not\equiv 0$, is by a factor of z^N and so an error term of $O(z^{(N+1)(k+1)})$ remains which is sufficient to show that the error in the reduced sequence can be made smaller than any arbitrary power of z, by going far enough. This result in turn implies the existence of an arbitrarily large number of diagonal types of integral polynomials. For if there were only a finite number of types, then one of them must make the integral polynomial defining equations accurate to within an error of only an arbitrarily high power of z, which means, in fact, that it would be a minimal integral polynomial representation for all higher order diagonal types and that is a contradiction to the assumption that there are only a finite number of diagonal types. (It does not, of course, preclude that they may all degenerate to be the same actual polynomials.) Therefore we can conclude directly, since the above is true for arbitrary N, that:

Theorem 3.1. *There exists at least an infinite subsequence of minimal, diagonal, integral polynomials to a given functional element $f(z)$ such that $\vec{Q}_N(0) \neq 0$.*

This property will be seen to be important later when the accuracy-through-order property of integral approximants is discussed.

We now introduce two more definitions which specify sequences of greater generality than the diagonal sequence. Definition 3.2 encompasses the kinds of sequences for which convergence theorems might possibly be proven. Definition 3.3 specifies a fairly general class of sequences which includes the natural generalizations of a staircase sequence of Padé approximants, as well as the diagonal and partially diagonal sequences. Theorem 3.4 establishes that at least an infinite subsequence of any of these sequences of polynomials do not all vanish at the origin.

Definition 3.2. *A partially diagonal sequence of integral polynomials will be any sequence where the elements of a given subset, $\mathcal{N} \subset \mathcal{I} = \{L, m_i,\ i = 0, \dots, k\}$ are set equal to the sequence index N and the rest, $\mathcal{I} \setminus \mathcal{N} \not\equiv \emptyset$, remain fixed.*

Definition 3.3. *A sequence $\{L_i, \vec{m}^{(i)}\}_{i=0}^{\infty}$ of integral polynomials is said to be a <u>connected, increasing</u> sequence, if both*

$$\begin{aligned} L_i &\geq L_{i-1}, \\ m_j^{(i)} &\geq m_j^{(i-1)},\ j = 0, 1, \dots, k, \end{aligned} \tag{3.5}$$

and

$$\max_{j=0,\dots,k} \{L_i - L_{i-1}, m_j^{(i)} - m_j^{(i-1)}\} = 1 \tag{3.6}$$

for $i = 1, 2, 3, \dots$.

An integral polynomial of type $(L, \vec{M})$ is called sub-minimal when it is a minimal integral polynomial of type $(l, \vec{m}) \leq (L, \vec{M})$ with strict inequality in at least one of the degrees.

Theorem 3.4. *Any connected, increasing sequence of integral polynomials contains an infinite subsequence of minimal or sub-minimal integral polynomials $\{\tilde{P}_{L_i}, \tilde{\vec{Q}}_{\vec{m}^{(i)}}\}_{i=0}^{\infty}$ to a given functional element such that $\tilde{\vec{Q}}_{\vec{m}^{(i)}}(0) \neq 0$.*

Proof. For each index set $(L_i, \vec{m}^{(i)})$, let $\left(P_{L_i}(z), \vec{Q}_{\vec{m}^{(i)}}(z)\right)$ denote a corresponding set of $k+2$ non-trivial integral polynomials. With M_i defined by

$$M_i = \sum_{j=0}^{k} (m_j^{(i)} + 1) - 1,$$

we note that these integral polynomials are constructed to satisfy

$$Q_{m_k^{(i)}}(z) f^{(k)}(z) + \cdots + Q_{m_0^{(i)}}(z) f(z) - P_{L_i}(z) = O(z^{L_i + M_i + 1}). \tag{3.7}$$

Let α_i be the highest power of z such that z^{α_i} factors all $k+2$ polynomials, and for $i = 0, 1, 2, \dots$, let

$$\begin{aligned} \beta_i &= i - \alpha_i, \\ \hat{Q}_{\vec{m}^{(i)}}(z) &= z^{-\alpha_i} Q_{\vec{m}^{(i)}}(z), \\ \hat{P}_{L_i}(z) &= z^{-\alpha_i} P_{L_i}(z), \\ M_{\beta_i} &= \sum_{j=0}^{k} (m_j^{(\beta_i)} + 1) - 1. \end{aligned} \tag{3.8}$$

From (3.7), it follows that

$$\hat{Q}_{m_k^{(i)}}(z) f^{(k)}(z) + \cdots + \hat{Q}_{m_0^{(i)}}(z) f(z) - \hat{P}_{L_i}(z) = O(z^{L_i+M_i-\alpha_i+1}). \tag{3.9}$$

The second phase of the proof consists of showing that $\left(\hat{P}_{L_i}(z), \hat{\vec{Q}}_{\vec{m}^{(i)}}(z)\right)$ are integral polynomials of type $\left(L_{\beta_i}, \vec{m}^{(\beta_i)}\right)$. From (3.6), we have

$$\begin{aligned} m_j^{(i)} - m_j^{(\beta_i)} &\leq i - \beta_i, \ j = 0, 1, \ldots, k, \\ L_i - L_{\beta_i} &\leq i - \beta_i, \end{aligned}$$

for each i, and hence, by (3.8),

$$\begin{aligned} m_j^{(i)} - \alpha_i &\leq m_j^{(\beta_i)}, \ j = 0, 1, \ldots, k, \\ L_i - \alpha_i &\leq L_{\beta_i}. \end{aligned}$$

Therefore the degrees of $\left(\hat{P}_{L_i}(z), \hat{\vec{Q}}_{\vec{m}^{(i)}}(z)\right)$ are compatible with their being of type $\left(L_{\beta_i}, \vec{m}^{(\beta_i)}\right)$. From (3.6), we have

$$L_i - L_{\beta_i} + \sum_{j=0}^{k} (m_j^{(i)} - m_j^{(\beta_i)}) \geq i - \beta_i,$$

and so, by (3.8),

$$L_i + M_i - \alpha_i + 1 \geq L_{\beta_i} + M_{\beta_i} + 1.$$

Therefore the order of contact in (3.9) suffices for $\left(\hat{P}_{L_i}, \hat{\vec{Q}}_{\vec{m}^{(i)}}(z)\right)$ to be integral polynomials of type $\left(L_{\beta_i}, \vec{m}^{(\beta_i)}\right)$. By construction, they have the property that

$$\hat{Q}_{m_j^{(i)}}(0) \neq 0,$$

for each i and a corresponding value of j. The minimality property is established in the same way as it was for Theorem 3.1, but only with respect to type $(L_i - \alpha_i, \overrightarrow{m^{(i)} - \alpha_i}) \leq (L_{\beta_i}, \vec{m}^{(\beta_i)})$ and so the integral polynomials are at least sub-minimal. ∎

Corollary 3.5. *There exists at least an infinite subsequence of minimal or sub-minimal, partially diagonal, integral polynomials to a given functional element $f(z)$ such that $\vec{Q}_L(0) \neq 0$.*

Proof. A special case of the previous result. ∎

We are now in a position to make some remarks about the structure of the set of integral approximants to a given functional element. Suppose that the approximant of type $(\lambda, \vec{\mu})$ has the true degree of all the polynomials equal to their nominal degree and $Q_{\mu_j}(0) \neq 0$ for some $0 \leq j \leq k$. Suppose further that the integral polynomial defining equations are over-satisfied by $r > 0$ powers of z. Then we can add, for example, $0 \cdot z^{\mu_i+1}$ to $Q_{\mu_i}(z)$ and have a solution for the $(\lambda, \mu_0, \ldots, \mu_{i-1}, \mu_i+1, \mu_{i+1}, \ldots, \mu_k)$-type integral polynomials which is identical to the $(\lambda, \vec{\mu})$ integral polynomials. In fact, by the same line of reasoning, it follows directly that the $(\lambda, \vec{\mu})$ integral polynomials fill the simplex with vertex at $\lambda, \vec{\mu}$ consistent with the required degree of contact. In addition, if $r \geq k+2$, then the polynomials $zP_\lambda(z), z\vec{Q}_\lambda(z)$ are the integral polynomials where every index has been advanced by unity and the integral polynomial defining equations are over-satisfied by $r-k-1$ extra powers of z. Thus the we can fill a new simplex with these polynomials with its vertex displaced from the old one by increasing each index by unity. This new simplex will partially fill the "$r+1$st layer" measured from the vertex of the original simplex, and lying outside of it. This process can be continued until the over-satisfaction of the integral polynomial equations is reduced to less than $k+2$, and defines the region in index space that is ocuppied by solutions of the integral polynomial defining equations which all correspond to the same integral approximants (a common factor in the polynomials will not change the approximants).

Let us also consider the previous class of degeneracies from another perspective. As we stated in Section 2, the derivation of the integral polynomials $P_L(z)$, $\vec{Q}_L(z)$ of given type $(L,\vec{M})$ is most directly based on the solution of a set of linear equations. We express the M equations determining $\vec{Q}_L(z)$ in the form,

$$A\,\underset{\rightarrow}{q} = \underset{\rightarrow}{0}, \tag{3.10}$$

where $\underset{\rightarrow}{q}$ is the vector containing the $M+1$ coefficients of all the $Q_{0,L}(z), Q_{1,L}(z), \ldots, Q_{k,L}(z)$ polynomials. The case we have just discussed, in which $\left(P_\lambda(z), \vec{Q}_\lambda(z)\right)$ and $\left(zP_\lambda(z), z\vec{Q}_\lambda(z)\right)$ are both integral polynominals of the same specified type, is an example of a situation in which the matrix A necessarily has rank less than M. To be specific, we will suppose that A has rank $M-n$. Then we may choose any n elements of the solution vector $\underset{\rightarrow}{q}$ at will. In particular, for any integers $\mu_j \geq 0$, satisfying $\mu_1+\mu_2+\cdots+\mu_k = n$, we can arrange that

$$\partial\{Q_{j,L}(z)\} \leq m_j - \mu_j, \quad j = 0,1,\ldots,k, \tag{3.11}$$

where $\partial\{\,\}$ denotes the true degree of the polynomial. In this way, we exhibit one family of solutions which will include the minimal one. Another "natural family" can be defined by replacing the degree conditions (3.11) by mixed degree and order conditions,

$$\partial\{Q_{j,L}(z)\} \leq m_j,\ Q_{j,L}(z) = o(z^{\mu_j}),\ j = 0,1,\ldots,k,$$

with the same conditions on μ_j. Despite the large number of different ways in which the integral polynomials could be specified, the algebraic equations (3.10) show that fundamentally there are exactly n+1, linearly independent solutions in the cases considered here. In the sense of polynomials, as we showed in Theorem 2.1, there are $\max(n+1,k+1)$ linearly independent solutions.

In light of the results in this section, it seems reasonable to require $\vec{Q}_L(0) \neq 0$ as part of our definition, as we will not eliminate any essentially different, minimal, integral approximants thereby. Although the behavior of the integral polynomials is well described, the problem of the behavior of the approximants in the non-minimal cases is harder and we have not explored it.

Lastly, we observe that there is no reason why the subsequence of sets of integral polynomials whose existence has been established in this section should all be distinct. Indeed, consideration of the case where they are essentially the same is the basis of the next result.

4. Equivalence Theorem

In this section we will be concerned with the proof of an equivalence theorem of the Kronecker type.

Theorem 4.1. *The statement: (i) $f(z)$ is a functional element at $z=0$ which satisfies*

$$Q_{k,m_k}(z)f^{(k)}(z)+Q_{k-1,m_{k-1}}(z)f^{(k-1)}(z)+\cdots+Q_{0,m_0}(z)f(z)-P_l(z)=0, \tag{4.1}$$

with $(P_l,\vec{Q}_{\vec{m}})$ minimal, is equivalent to: (ii) $g(z)$ is a functional element at $z=0$ for which

$$(P_l,\vec{Q}_{\vec{m}})=\lambda_{L,\vec{M}}(P_L,\vec{Q}_{\vec{M}})\ \forall\,(L,\vec{M})\geq(l,\vec{m}), \tag{4.2}$$

where $(P_L,\vec{Q}_{\vec{M}})$ are the integral polynomials to $g(z)$, $\lambda_{L,\vec{M}}\neq 0$ is a constant factor which can be normalized to $\lambda_{L,\vec{M}}=1$ and $g(z)=f(z)$ of (i).

The subscript notation is this section varies slightly from the usages in the previous sections.

Proof. First, (i) implies (ii). Consider any $(L,\vec{M})\geq(l,\vec{m})$. By eq. (4.1) $(P_l,\vec{Q}_{\vec{m}})$ is a solution of the integral polynomial defining equation for type $(L,\vec{M})$, but $(P_l,\vec{Q}_{\vec{m}})$ is minimal by hypothesis. Therefore we can conclude (4.2), except for the $f(z)=g(z)$ statement which we will discuss below.

Second we must consider, does (ii) imply (i)? Thus let us now suppose that (ii) holds. Therefore we must have,

$$Q_{k,m_k}(z)g^{(k)}(z)+\cdots+Q_{0,m_0}(z)g(z)-P_l(z)=O(z^t),\ \forall t<\infty. \tag{4.3}$$

By the minimality properties, $P_l,\vec{Q}_{\vec{m}}$ have no common, non-constant factors like $(1+az)$. As we saw in the previous section, if we divide out any factors of z^j which may occur, (4.3) does not change, except that $m_i\to m_i-j$, $l\to l-j$, as it holds for all $t<\infty$. Since $g(z)$ is a functional element, the left side of (4.3) is regular at $z=0$ and so, by (4.3) must be identically zero. Thus, except for the $f(z)=g(z)$ statement, (ii) implies (i), by analyticity.

The problem that remains to complete the proof is one of boundary conditions. Specifically we need to show that there exists a finite N such that there is a subset of $\{f^{(i)}(0),\ i=0,1,\ldots,N\}$ which suffices to determine the solution of (4.1) uniquely. Now it can't be in (4.3) that $Q_j(0)=0,\ j=0,\ldots,k$ and $P_l(0)\neq 0$, since, $g^{(j)}(0)<\infty,\ j=0,\ldots,k$ and (4.3), would imply $P_l(0)=0$, a contradiction. Therefore at least for some $j_0, 0\leq j_0\leq k,\ Q_{j_0}(0)\neq 0$.

Next define γ as

$$\gamma=\max(k-\alpha_k,k-1-\alpha_{k-1},\ldots,-\alpha_0), \tag{4.4}$$

where $Q_{k,m_k}(z)\propto z^{\alpha_k}$ as $z\to 0$, and, of course, we necessarily have $\alpha_k\leq m_k$. Also by our construction, $\alpha_{j_0}=0$. We can therefore immediately conclude that

$$\gamma\geq j_0\geq 0. \tag{4.5}$$

Now select r as the smallest integer such that

$$\begin{aligned}
[z^{\gamma-k}Q_k(z)]_{z=0} &= [z^{\gamma-k+1}Q_{k-1}(z)]_{z=0} = \\
\cdots\qquad &= [z^{\gamma-k+r-1}Q_{k-r+1}(z)]_{z=0}=0, \\
[z^{g-k+r}Q_{k-r}(z)]_{z=0}\neq 0, &
\end{aligned} \tag{4.6}$$

where by (4.4) no negative powers of z occur in (4.6). Because $g(z)$ is assumed to be a functional element, we can write $g(z)=z^\rho\phi(z)$, where $\rho\geq 0$ is an integer and

$$\phi(z)=\sum_{j=0}^{\infty}c_jz^j,\ c_0\neq 0. \tag{4.7}$$

In order to analyze the solution of (4.3) [or (4.1)] we can consider the indicial equation, but first we need a little more notation. If we multiply (4.3) by z^γ, we get, using (4.7),

$$Q_{k,m_k}(z)z^{\gamma+\rho}\sum_{j=0}^{\infty}c_j[\rho+j]_k z^{j-k}+\cdots+Q_{0,m_0}(z)z^{\gamma+\rho}\sum_{j=0}^{\infty}c_j z^j - z^\gamma P_l(z) = O(z^{t+\gamma})$$
$$\Rightarrow\ = O(z^t), \tag{4.8}$$

as t is arbitrary, where $[\rho+j]_k \equiv (\rho+j)(\rho+j-1)\cdots(\rho+j-k+1)$, and $[\rho+j]_0 = 1$. From (4.8), the coefficient of $c_j z^{\rho+j}$ will be

$$\begin{aligned} f(z,\rho+j) &= Q_{k,m_k}(z)z^{\gamma-k}[\rho+j]_k + Q_{k-1,m_{k-1}}(z)z^{\gamma-k+1}[\rho+j]_{k-1}+\cdots+z^\gamma Q_{0,m_0}(z) \\ &= \sum_{\lambda=0}^{\max_k(m_k+\gamma-k)} f_\lambda(\rho+j)z^\lambda, \end{aligned} \tag{4.9}$$

and so defining $f_\lambda(\rho+j)$ as the coefficients of z^λ in the central expression of (4.9). Therefore we may rewrite (4.8) as

$$\sum_{j=0}^{\infty} z^{\rho+j}[c_j f_0(\rho+j)+c_{j-1}f_1(\rho+j-1)+\cdots+c_0 f_j(\rho)] - z^\gamma P_l(z) = O(z^t). \tag{4.10}$$

It is to be noted that the number of terms in [] in (4.10) is limited by the restriction noted in (4.9). Now the lowest power of z comes from the $j=0$ term and it is contained in

$$z^\rho c_0 f(\rho) - z^\gamma P_l(z) = 0. \tag{4.11}$$

The P_l term may or may not contribute depending on the values of γ and ρ. First let us consider the possibility that $f_0(\rho)=0$. By (4.9) it is

$$\left[z^\gamma\left(Q_{k,m_k}(z)[\rho]_k+\cdots+Q_{0,m_0}(z)\right)\right]_{z=0} = 0. \tag{4.12}$$

Now by the way that we have defined r in (4.6), we see that (4.12) becomes

$$\left[z^{\gamma-k+r}Q_{k-r}(z)\right]_{z=0}[\rho]_{k-r}+\cdots = 0, \tag{4.13}$$

so that we have a polynomial in ρ of degree $k-r$ exactly. If $r=0$, then the origin is a regular singular point. If $r<k$, then $f_0(\rho)$ is in fact a polynomial in ρ and is not identically zero. On the other hand, if $r=k$, Ince [23] has proven that there do not exist any regular solutions of (4.3), but $g(z)$ is a regular solution, therefore, $r<k$, and $f_0(\rho)\not\equiv 0$.

We are only concerned with those roots which are integers and which obey

$$\rho \ge \rho_{g(z)}, \tag{4.14}$$

where $\rho_{g(z)}$ is that ρ defined by $g(z)$ through (4.7). Now, referring to (4.11), we see that there are several cases to consider. In case (1), $\rho<\gamma$, the indicial equation from (4.11) becomes

$$c_0 f_0(\rho) = 0,\ \Rightarrow\ f_0(\rho)=0 \text{ as } c_0\neq 0. \tag{4.15}$$

If we use the notation,

$$P_l(z) = \sum_{i=0}^{l} p_i z^i, \text{ and define } p_i = 0 \text{ if } i < 0 \text{ or } i > l, \tag{4.16}$$

then the recursion relations for the coefficients in (4.7) are

$$\begin{aligned} &c_0 f_0(\rho) - p_{\rho-\gamma} = 0, \\ &c_1 f_0(\rho+1) + c_0 f_1(\rho) - p_{\rho-\gamma+1} = 0, \\ &\quad\vdots \\ &c_j f_0(\rho+j) + \cdots + c_0 f_j(\rho) - p_{\rho-\gamma+j} = 0, \\ &\quad\vdots \end{aligned} \tag{4.17}$$

Thus to specify all the c_j we need c_0 and up to $k-r-1$ more c_j which correspond to the integer roots of $f_0(\rho+j)$ which satisfy the restriction (4.14).

In the simple case, where all the $\alpha_j = 0,\ 0 \le j \le k$,

$$f_0(\rho) = [\rho]_k, \tag{4.18}$$

and has zeros for $\rho = 0, 1, \ldots, k-1$ so we get the standard boundary condition requirement that $c_0, c_1, \ldots, c_{k-1}$ determine (together, of course, with the differential equation) the explicit unique solution.

In general, in this case, the required set of boundary conditions is an explicit set of $\le k$ c_j's derived from the structure of the differential equation for $g(z)$. Thus the equation plus known boundary conditions give an explicit construction of a unique (in the sense of analytic continuation) functional element. But g(z) is such an element and the equation is minimal, therefore in case (1) $f(z) = g(z)$, so (ii) implies (i).

Now consider case (2) where $\rho > \gamma + l$. Again, $P_l(z)$ does not contribute to the indicial equation, so the result is the same here as in case (1) and again we conclude that (ii) implies (i).

Finally, in case (3) $\gamma \le \rho \le \gamma + l$. The indicial equation becomes, from (4.11),

$$f_0(\rho) - p_{\rho-\gamma}/c_0 = 0. \tag{4.19}$$

This equation is, of course, satisfied by hypothesis, by the ρ, c_0 which come from $g(z)$. There are at most $k - r \le k$ additional possible integer zeros of $f_0(\rho+j)$. If $r \ge 1$, then the assignment of at most k c_j's suffices to determine the solution uniquely, and it must be $g(z)$.

If $r = 0$, then there is the possibility that all k solutions of the differential equation are regular and non-singular. As $p_{\rho-\gamma}/c_0 \neq 0$ [if it were then we would be in either case (1) or case (2)] there can be k roots of $f_0(\rho) = 0$ and so we would seem to need up to k more c_j's besides c_0, whereas there are only, at most, k linearly independent solutions of the differential equation. However, the boundary conditions always implicitly specify ρ and in this case, ρ implies c_0 by eq. (4.19), so it is not an independent constant. Therefore, we really need at most only k c_j's and thus again (ii) implies (i). ■

An interesting example is relevant at this point. It shows that in general, in contrast to the Padé approximant case, the series coefficients which suffice to determine uniquely the integral polynomials of a particular type may not determine the integral approximant uniquely. Consider $f(z)$ determined by the equation,

$$zf'' - 9f' + zf = 2z. \tag{4.20}$$

The solution is necessarily an even function of z, and the indicial equation is

$$\rho(\rho-10)=0, \tag{4.21}$$

provided $\rho = 2$ is not a root, which it is not. The $[1/1;0;1]$ approximant requires the series coefficients through order z^5 in f'' or equivalently through order z^7 in $f(z)$. Nevertheless, by (4.17), since $f_0(10)=0$, the construction of a unique solution for the approximant requires a knowledge of c_{10}. However the series for the solution of the differential equation through $O(z^9)$ may still be determined uniquely from $c_0 = 1$ and is, for reference,

$$f(z) = 1 - \frac{z^2}{16} - \frac{z^4}{384} - \frac{z^6}{9216} - \frac{z^8}{147456} + O(z^{10}) \tag{4.22}$$

It can be verified by an easy computation that this section of the Taylor series also determines the $[1/1;0;1]$ integral polynomials uniquely, once some coefficient is fixed, *e.g.* $Q'_{2,1}=1$. The nature of the theorem we have proven in this section is such that we do not need to confront this source of possible non-uniqueness as assumptions are made concerning all the series coefficients.

5. Accuracy through Order Properties of Integral Approximants

As a reminder, in the case of Padé approximants, the defining equations

$$Q_M(z)f(z) - P_L(z) = O(z^{L+M+1}),\ Q_M(0)=1, \tag{5.1}$$

may readily be solved to yield an approximant to $f(z)$ with the accuracy,

$$f(z) - \frac{P_L(z)}{Q_M(z)} = O(z^{L+M+1}). \tag{5.2}$$

The situation in the integral approximant case is more complicated. The given functional element $f(z)$ satisfies the equation

$$Q_{k,m_k}(z)f^{(k)}(z) + \cdots + Q_{0,m_0}(z)f(z) - P_L(z) \equiv r(z) = O(z^{L+M+1}), \tag{5.3}$$

in the notation of (2.1), which defines the remainder term $r(z)$. The integral approximant is the solution of

$$Q_{k,m_k}(z)y^{(k)}(z) + \cdots + Q_{0,m_0}(z)y(z) - P_L(z) = 0. \tag{5.4}$$

The difference or error term, $d(z) = f(z) - y(z)$ satisfies

$$Q_{k,m_k}(z)d^{(k)}(z) + \cdots + Q_{0,m_0}(z)d(z) = r(z). \tag{5.5}$$

Let $\phi_\nu(z),\ \nu = 1,\ldots,k$ be a set of principal solutions of the homogeneous version of (5.5). For our discussion we will assume that $z=0$ is at worst a regular singular point. In terms of the ϕ_ν's the solution of the non-homogeneous eq. (5.5) is (Kamke [24])

$$d(z) = \sum_{\nu=1}^{k} \phi_\nu(z) \int^z \frac{W_\nu(\zeta)d\zeta}{Q_{k,m_k}(\zeta)W(\zeta)}\ (k\ge 1), \tag{5.6}$$

where $W(\zeta)$ is the Wronskian,

$$W(\zeta) = \det \begin{vmatrix} \phi_1(\zeta) & \dots & \phi_k(\zeta) \\ \phi_1'(\zeta) & \dots & \phi_k'(\zeta) \\ \vdots & \ddots & \vdots \\ \phi^{(k-1)}(\zeta) & \dots & \phi_k^{(k-1)}(\zeta) \end{vmatrix}, \tag{5.7}$$

and $W_\nu(\zeta)$ replaces the νth column of W by

$$\begin{pmatrix} 0 \\ \vdots \\ 0 \\ r(\zeta) \end{pmatrix}. \tag{5.8}$$

If every principal solution ϕ_ν is regular at $z = 0$, then $Q_{k,m_k}(0) \neq 0$ which implies that $W(0) \neq 0$ as by Abel's formula,

$$W(\zeta) \propto \exp\left(- \int^{\zeta} \frac{Q_{k-1,m_{k-1}}(\xi) d\xi}{Q_{k,m_k}(\xi)} \right), \tag{5.9}$$

where $\propto$ denotes proportional to as usual. Also in this case $W_\nu(\zeta)$ is finite and at $\zeta = 0$ and proportional to ζ^{L+M+1} which in turn, by (5.6), implies that

$$d(z) \propto O(z^{L+M+2}),\ k \geq 1, \tag{5.10}$$

so in this case the solution preserves the accuracy-through-order of the remainder term in the integral polynomial determining equations and even improves it slightly. [The case $k = 0$ is treated in (5.2).]

If instead of all the principal solutions being regular at $z = 0$, they all possess regular singularities (without logarithms which only mess up the discussion and don't dramatically change the result), then we have

$$\phi_\nu(z) \propto z^{r_\nu},\ \text{as } z \to 0. \tag{5.11}$$

(Of course, it can not really be this way as at least one of the solutions must be regular by hypothesis, but we will discuss an intermediate case below which can encompass the actual one.) In this case,

$$W(\zeta) \propto \zeta^{\left[\sum_{i=1}^{k} r_k - 0.5k(k+1)\right]}, \tag{5.12}$$

and

$$\phi_\nu(z) W_\nu(\zeta) \propto \left(\frac{z}{\zeta}\right)^{r_\nu} \zeta^{\left[\sum_{i=1}^{k} r_i - 0.5(k-1)(k-2)\right]}. \tag{5.13}$$

So near $z = 0$,

$$\begin{aligned} d(z) &\propto \sum_{\nu=1}^{k} \int^{z} \frac{r(\zeta)(z/\zeta)^{r_\nu} \zeta^{k-1} d\zeta}{Q_{k,m_k}(\zeta)} \\ &\propto O\left(z^{L+M+1+k-\alpha_k}\right) \end{aligned} \tag{5.14}$$

But if we have all regular singular solutions (and z^j is not a factor of $P, \vec{Q}$ as we have discussed in previous sections), then $\alpha_k = k$. Therefore,

$$d(z) \propto O(z^{L+M+1}), \tag{5.15}$$

and the accuracy-through-order property is again preserved.

For the cases where there are some regular solutions and the rest of the solutions correspond to regular singularities, Abel's formula still gives the Wronskian as a power of ζ near $\zeta = 0$, and it also can give $W_\nu(\zeta)$ as the Wronskian of a reduced differential equation of order $k-1$. The result is a formula much like (5.14) but with a few minor modifications, which do not particularly affect the order of the error term.

We have not analyzed the accuracy through order properties at irregular singular points. Certainly it is an exceptional case even for a regular solution (known by hypothesis to occur for our case) to occur at an irregular singular point.

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